

CURVATURE-DIRECTION MEASURES OF SELF-SIMILAR SETS

TILMAN JOHANNES ROTHE AND MARTINA ZÄHLE

ABSTRACT. We obtain fractal Lipschitz-Killing curvature-direction measures for a large class of self-similar sets F in \mathbb{R}^d . Such measures jointly describe the distribution of normal vectors and localize curvature by analogues of the higher order mean curvatures of differentiable submanifolds. They decouple as independent products of the unit Hausdorff measure on F and a self-similar fibre measure on the sphere, which can be computed by an integral formula. The corresponding local density approach uses an ergodic dynamical system formed by extending the code space shift by a subgroup of the orthogonal group. We then give a remarkably simple proof for the resulting measure version under minimal assumptions.

1. INTRODUCTION

The "second order" anisotropic structure of self-similar sets F in \mathbb{R}^d is studied by means of approximation with parallel sets $F(\epsilon)$ of small distances ϵ . This leads to fractal curvature-direction measures and their local "densities". From the isotropic point of view this was first investigated in the pioneering work by Winter [Win08] (deterministic self-similar sets with polyconvex parallel sets, curvature measures) and then in [Zäh11] (self-similar random sets with singular parallel sets, total curvatures), [WZ10] (deterministic measure version for singular parallel sets) and [RZ10] (dynamical approach to curvatures measures and their local densities). The special case of the Minkowski content was treated earlier, e.g. in [LP93] and [Fal95] ($d = 1$), in [Gat00] (self-similar random fractals for any d), in [KK10] ($d = 1$ self-conformal sets), and in a very general context recently in [RW09].

In the present paper we extend these results to anisotropic quantities for the fractal sets (cf. the remark at the end of the paper). We mainly follow the dynamical approach from [RZ10] and give a new and short proof for convergence of the corresponding measures under weaker assumptions, which considerably simplifies the former approaches.

The classical geometric background are extensions of Federer's curvature measures for sets of positive reach ([Fed59]) regarding normal directions, the so-called curvature-direction measures or generalized curvature measures (cf. [Sch80], [Zäh86], and various subsequent papers). In the context of differential geometry they would correspond to the integrals of higher order mean curvatures over those points of the boundary which together with the unit normal belong to a fixed Borel set in $\mathbb{R}^d \times S^{d-1}$. Since the boundaries under consideration are non-smooth, we are working with integrals of generalized mean curvatures over their unit normal bundles in the sense of [RZ05].

2000 *Mathematics Subject Classification.* Primary: 28A80, 28A75, 37A99 Secondary: 28A78, 53C65.

Key words and phrases. self-similar set, Lipschitz-Killing curvature-direction measure, fractal curvature measure, Minkowski content.

Both supported by grant DFG ZA 242/5-1.

Under some regularity condition on the parallel sets, which is always fulfilled if $d \leq 3$ (according to a result in [Fu85]), these signed measures $C_k(F(\epsilon), \cdot)$ on $\mathbb{R}^d \times S^{d-1}$ of order k are determined for almost all ϵ (see Section 2.2). Let G be the subgroup generated by the orthogonal components of the similarities associated with F . We first restrict the curvature-direction measures to covariant sets $A_F(x, \epsilon) \times \phi R$ from a suitable neighborhood net in $\mathbb{R}^d \times S^{d-1}$, e.g. to $(F(\epsilon) \cap B(x, a\epsilon)) \times \phi R$, where $a > 1$ is fixed, $x \in F$, R is a Borel set in S^{d-1} and ϕ an element of G . Under some integrability condition we obtain for any R and almost all $(x, \phi) \in F \times G$ an average limit of the rescaled versions

$$\epsilon^{-k} C_k(F(\epsilon), A_F(x, \epsilon) \times \phi R)$$

as $\epsilon \rightarrow 0$ (Theorem 3.5). For the total value at $R = S^{d-1}$ this corresponds to the k -th *fractal curvature densities* introduced in [RZ10]. As function in R the limits may be considered as second order fibre measures which, divided by the total values, describe the *local fractal direction 'distributions' weighted by curvatures*. Due to self-similarity and the behavior of the classical curvature-direction measures these limits do not depend on the base points x and on ϕ , i.e., we obtain at almost all points constant fractal curvature densities, constant fibre measures and therefore also constant local curvature-direction distributions. In this part of our approach an extended ergodic dynamical system together with the geometric scaling and invariance properties of F , A_F and C_k provide the main tools.

Then an appropriate choice of the neighborhood net A_F , similar as in [RZ10], and a new method of proof is used in order to derive the corresponding measure result: Under some slightly stronger uniform integrability condition the rescaled curvature-direction measures

$$\epsilon^{D-k} C_k(F(\epsilon), \cdot)$$

weakly converge in the average as $\epsilon \rightarrow 0$ to a *fractal curvature-direction measure* which is the product of the normalized Hausdorff measure \mathcal{H}^D on F and the above fibre measure (Theorem 3.9).

(The fractal curvature measures of order $k = 0, 1, \dots, d$ from [RZ10] arise as the projection measures on the base point component. Because of self-similarity they are all constant multiples of the normalized Hausdorff measure on F and the constants agree with the above fractal curvature densities.)

The new approach makes the proof much shorter, even in our non-isotropic version. It enlightens the essential measure theoretic background and shows that the uniform integrability condition is sharp.

Finally, a modified Sierpinski gasket is discussed as an example.

2. BASIC NOTIONS

2.1. Self-similar sets, code space, and measures. The notion of self-similar sets is well-known from the literature. See Hutchinson [Hut81] for the first general approach and the relationships mentioned below without a reference. We use here the following notation and results.

The basic space is a compact set $J \subset \mathbb{R}^d$ with $J = \overline{\text{int } J}$. S_1, \dots, S_N denotes the generating set of *contracting similarities* in \mathbb{R}^d with *contraction ratios* r_1, \dots, r_N and orthogonal group components $\varphi_1, \dots, \varphi_N$.

We assume the *strong open set condition* (briefly (SOSC)) with respect to $\text{int } J$, i.e.,

$$\bigcup_{j=1}^N S_j(J) \subset J, \quad S_j(\text{int } J) \cap S_l(\text{int } J) = \emptyset, \quad j \neq l,$$

and that there exists a sequence of indices $l_1, l_2, \dots, l_m \in \{1, \dots, N\}$ such that

$$S_{l_1} \circ S_{l_2} \dots \circ S_{l_m}(J) \cap \text{int } J \neq \emptyset.$$

The latter (strong) condition is here equivalent to

$$F \cap \text{int } J \neq \emptyset$$

where F denotes the associated self-similar fractal set F . (According to a result of Schief (see [Sch94]), (SOSC) for some J is already implied by the open set condition on the similarities. A characterization of (SOSC) in algebraic terms of the S_i is given in Bandt and Graf [BG92].) The set F may be constructed by means of the *code space* $W := \{1, \dots, N\}^{\mathbb{N}}$, the set of all infinite words over the alphabet $\{1, \dots, N\}$. We write $W^n := \{1, \dots, N\}^n$ for the set of all words of length $|w| = n$, $W^* := \bigcup_{n=1}^{\infty} W^n$ for the set of all finite words, $w|n := w_1 w_2 \dots w_n$ if $w = w_1 w_2 \dots w_n, w_{n+1} \dots$ for the *restriction* of a (finite or infinite) word to the first n components, and vw for the *concatenation* of a finite word v and a word w . If $w = w_1 \dots w_n \in W^n$ we also use the abbreviations $S_w := S_{w_1} \circ \dots \circ S_{w_n}$, $\varphi_w := \varphi_{w_1} \circ \dots \circ \varphi_{w_n}$, and $r_w := r_{w_1} \dots r_{w_n}$ for the contraction ratio of this mapping. Finally we denote $K_w := S_w(K)$ for any compact set K and $w \in W^*$. (For completeness we also write $K_{\emptyset} := K$.) In these terms the set F is determined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{w \in W^n} J_w$$

and characterized by the *self-similarity property* $F = S_1(F) \cup \dots \cup S_N(F)$. Iterated applications yield

$$F = \bigcup_{w \in W^n} F_w, \quad n \in \mathbb{N}.$$

As in the literature, we will use the abbreviation

$$S(K) := \bigcup_{j=1}^N S_j(K)$$

for compact sets K , i.e., $F = S^n(F)$, $n \in \mathbb{N}$.

Alternatively, the self-similar fractal F is the image of the code space W under the *projection* π given by

$$\pi(w) := \lim_{n \rightarrow \infty} S_{w|n} x_0$$

for an arbitrary starting point x_0 . The mapping $w \mapsto x = \pi(w)$ is biunique except for a set of points x of D -dimensional Hausdorff measure \mathcal{H}^D zero, and the *Hausdorff dimension* D of F is determined by

$$(2.1.1) \quad \sum_{j=1}^N r_j^D = 1.$$

Up to exceptional points we *identify* $x \in F$ with its *coding sequence* and write $x_1 x_2 \dots$ for this infinite word, i.e. $\pi(x_1 x_2 \dots) = x$, and write

$$x|n := x_1 \dots x_n$$

for the corresponding finite words.

If ν denotes the infinite product measure on W determined by the probability measure on the alphabet $\{1, \dots, N\}$ with single probabilities r_1^D, \dots, r_N^D , then the normalized D -dimensional Hausdorff measure with support F equals

$$(2.1.2) \quad \mu_F := \mathcal{H}^D(F)^{-1} \mathcal{H}^D(F \cap (\cdot)) = \nu \circ \pi^{-1}.$$

It is also called the natural *self-similar measure* on F , since we have

$$(2.1.3) \quad \mu_F = \sum_{j=1}^N r_j^D \mu_F \circ S_j^{-1}.$$

G is the compact group generated by all the φ_j , i.e. the orthogonal group components of the S_j :

$$(2.1.4) \quad G := \text{cl} \{ \varphi_j : j = 1, \dots, N \} \subseteq O(d)$$

Denote its normalized Haar measure by \mathcal{H}_G .

From (SOSC) on J one obtains

$$(2.1.5) \quad \int |\ln d(y, J^c)| d\mu_F(y) < \infty$$

([Gra95, Proposition 3.4]) and $\mu_F(\partial J) = 0$. Furthermore, by the open set condition F is a D -set, i.e., there exist positive constants c_F and C_F such that

$$(2.1.6) \quad c_F r^D \leq \mathcal{H}^D(F \cap B(x, r)) \leq C_F r^D, \quad x \in F, \quad r \leq \text{diam } F.$$

Finally, we write $f \otimes g(x, n) = f(x)g(n)$ if $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : S^{-1} \rightarrow \mathbb{R}$, and $\mu \otimes \nu$ for the product measures of μ and ν , and $\mu(f)$ for $\int f d\mu$.

2.2. Curvature-direction measures of parallel sets. We will use the following notations for points x and subsets E of \mathbb{R}^d :

$$d(x, E) := \inf_{y \in E} |x - y|, \quad |E| := \text{diam } E = \sup_{x, y \in E} |x - y|.$$

The background from classical singular curvature theory is summarized in [Zäh11]. We recall some of those facts. For certain classes of compact sets $K \subset \mathbb{R}^d$ (including many classical geometric sets) it turns out that for Lebesgue-almost all distances $r > 0$ the parallel set (r -tube, offset, Minkowski sausage)

$$K(r) := \{x : d(x, K) \leq r\}$$

possesses the property that the closure of its complement

$$\widetilde{K(r)} := \overline{K(r)^c}$$

is a set of positive reach in the sense of Federer [Fed59] with Lipschitz boundary. A sufficient condition is that r is a regular value of the Euclidean distance function to K (see Fu [Fu85, Theorem 4.1] together with [RZ03, Proposition 3]). (In \mathbb{R}^2 and \mathbb{R}^3 this is fulfilled for all K , see [Fu85]) In this case both the sets $\widetilde{K(r)}$ and $K(r)$ are *Lipschitz d -manifolds of bounded curvature* in the sense of [RZ05], i.e., their k -th *Lipschitz-Killing curvature-direction measures*, $k = 0, 1, \dots, d-1$, are determined in this general context and agree with the classical versions in the special cases. Their marginal $C_k(K(r), \mathbb{R}^d \times \cdot)$ is known as area measure in convex geometry. Moreover, they satisfy

$$(2.2.1) \quad C_k(K(r), \cdot) = (-1)^{d-1-k} C_k(\widetilde{K(r)}, \rho(\cdot)),$$

where $\rho(x, n) = (x, -n)$ is the reflection on $\mathbb{R}^d \times S^{d-1}$. The $C_k(K(r), \cdot)$ are signed measures with finite *variation measures* $C_k^{\text{var}}(K(r), \cdot)$ and the explicit integral representations are reduced to [Zäh86] (cf. [RZ05, Theorem 3] for the general case).

$C_{d-1}(K(r), \cdot \times S^{d-1})$ agrees with one half of the $(d-1)$ -dimensional Hausdorff measure \mathcal{H}^{d-1} on the boundary $\partial K(r)$. For completeness we define $C_d(K(r), \cdot)$ as *Lebesgue measure restricted to $K(r)$* , times the unique rotation invariant probability measure on S^{d-1} . The *total measures (curvatures)* of $K(r)$ are denoted by

$$C_k(K(r)) := C_k(K(r), \mathbb{R}^d \times S^{d-1}), \quad k = 0, \dots, d.$$

By an associated Gauss-Bonnet theorem (see [RZ03, Theorems 2,3]) the *total Gauss curvature* $C_0(K(r))$ coincides with the *Euler-Poincaré characteristic* $\chi(K(r))$.

In the present paper only the following main properties of the curvature measures for such parallel sets will be used, $A \subseteq \mathbb{R}^d$, $R \subseteq S^{d-1}$ Borel:

The curvature measures are *motion invariant*, i.e.,

$$(2.2.2) \quad C_k(g(K(r)), gA \times \varphi_g R) = C_k(K(r), A \times R) \\ \text{for any Euclidean motion } g, \text{ orthogonal component } \varphi_g$$

they are *homogeneous of degree k* , i.e.,

$$(2.2.3) \quad C_k(\lambda K(r), (\lambda A) \times R) = \lambda^k C_k(K(r), A \times R), \quad \lambda > 0,$$

and locally determined, i.e.,

$$(2.2.4) \quad C_k(K(r), (\cdot) \cap O \times S^{d-1}) = C_k(K'(r'), (\cdot) \cap O \times S^{d-1})$$

for any open set $O \subset \mathbb{R}^d$ such that $K(r) \cap O = K'(r') \cap O$, where $K(r)$ and $K'(r')$ are both parallel sets where the closures of the complements have positive reach.

3. CURVATURE-DIRECTION MEASURES OF SELF-SIMILAR SETS

3.1. Covariant local neighborhood nets. Throughout this entire paper, we will assume:

Assumption 3.1. (*Regularity of parallel sets*) For Lebesgue almost all $\epsilon > 0$:

- (1) $\text{reach } \widetilde{F(\epsilon)} > 0$,
- (a) $\text{nor } F(\epsilon) \cap \rho \text{ nor } \widetilde{F(\epsilon)} = \emptyset$.

Here $\text{nor } X$ denotes the unit normal bundle of a set with positive reach (as subset of $\mathbb{R}^d \times S^{d-1}$, cf. the reference papers) and ρ the normal reflection $(x, n) \mapsto (x, -n)$. Curvature measures of $\widetilde{F(\epsilon)}$ exist under these conditions. Many classes of sets satisfy this requirement, e. g. in case $F(\epsilon)$ is polyconvex, or any set F in space dimension $d \leq 3$.

In order to refine the results from [RZ10] and [WZ10] for the anisotropic case we will consider the covariant local neighborhood nets $\{A_F(x, \epsilon) \subseteq \mathbb{R}^d \times S^{d-1} : x \in F, 0 < \epsilon < \epsilon_0\}$ from [RZ10] marked by direction sets $R \in \mathcal{B}(S^{d-1})$. As a main step we will show that for a certain constant b , the measures

$$(3.1.1) \quad \Delta_{F,x,\delta}(R) := \frac{1}{|\ln \delta|} \int_{\delta}^{d(x,J^c)/b} \epsilon^{-k} C_k(F(\epsilon), A_F(x, \epsilon) \times R) \epsilon^{-1} d\epsilon$$

on S^{d-1} converge weakly as $\delta \rightarrow 0$ for μ_F -almost all x (Corr. 3.10). The limit measure $D_{C_k^{\text{frac}}|F}$ is the directional component of the fractal curvature-direction measure $C_k^{\text{frac}}(F, \cdot)$

to be derived on $F \times S^{d-1}$. To this aim we first consider the sets $A_F(x, \epsilon) \times \phi(R)$ for a fixed $R \in \mathcal{B}(S^{d-1})$ and calculate the limit

$$\lim_{\delta \rightarrow 0} \Delta_{F, x, \delta}(\phi R) = D_{C_k^{\text{frac}}|F}(\phi R)$$

for μ_F -almost all $x \in F$ and \mathcal{H}_G -almost all $\phi \in G$ (Theorem 3.9). The main tools are to translate the problem into the language of an extended shift dynamical system, and Birkhoff's ergodic theorem.

Definition 3.2. Let constants $a > 1$ and $\epsilon_0 > 0$ be given, and denote

$$b := \max(2a, \epsilon_0^{-1}|J|).$$

A *covariant neighborhood net* in F is a family of measurable sets

$$\{A_F(x, \epsilon) \subseteq \mathbb{R}^d \times S^{d-1} : x \in F, 0 < \epsilon < \epsilon_0\}$$

satisfying the following three conditions.

- (1) $A_F(x, \epsilon) \subseteq (F(\epsilon) \cap B(x, a\epsilon))$,
- (2) $A_F(x, \epsilon) = S_j A_F(S_j^{-1}(x), r_j^{-1}\epsilon)$ if $1 \leq j \leq N$, $x \in F_j$, and $\epsilon < d(x, (S_j J)^c)/b$.
- (3) The indicator function $1_{A_F(x, \epsilon)}(z)$ is a measurable function of $(x, \epsilon, z) \in F \times (0, \epsilon_0) \times \mathbb{R}^d$.

(Note that $r_j^{-1}\epsilon < \epsilon_0$ in (2).)

Example 3.3. Two possible choices of A_F are:

$$(3.1.2) \quad A_F(x, \epsilon) := (F(\epsilon) \cap B(x, a\epsilon)), \quad \epsilon > 0,$$

$$(3.1.3) \quad A_F(x, \epsilon) := \{x' \in F(\epsilon) : |x - x'| \leq \rho_F(x', \epsilon)\}, \\ 0 < \epsilon < \epsilon_0 := \mathcal{H}^D(F)^{1/D}$$

where $\rho_F(z, \epsilon)$ is determined for $0 < \epsilon < \epsilon_0$ by the condition

$$\rho_F(x', \epsilon) = \min\{\rho : \mathcal{H}^D(F \cap B(x', \rho)) = \epsilon^D\}.$$

The choice (3.1.3) is important because of its close ties with the measure version of our main results, as discussed in section 3.4, and is made to match (3.4.6). The requirements of the definition are met, see [RZ10, Lemma 2.1.2].

3.2. Main result 1: fibre measures. Recall G from (2.1.4).

Notation 3.4. For Borel sets $R \subseteq S^{d-1}$, define

$$(3.2.1) \quad D_{C_k^{\text{frac}}|F}(R) := \frac{1}{\sum_{j=1}^N r_j^D |\ln r_j|} \int_{F \times G} \int_{d(x, (S_{x_1} J)^c)/b}^{d(x, J^c)/b} \epsilon^{-k} C_k(F(\epsilon), A_F(x, \epsilon) \times \phi(R)) \epsilon^{-1} d\epsilon \, d(\mu_F \otimes \mathcal{H}_G)(x, \phi)$$

provided the integral exists. It serves as a limit object, and we will show it is a (signed) measure. $D_{C_k^{\text{frac}}|F}^\pm$ is defined by substituting the variation measure C_k^\pm for C_k in the formula.

The case of an infinite limit is only relevant to prove our results are sharp.

Theorem 3.5. (*Fibre measure*) Suppose that the self-similar set F in \mathbb{R}^d with Hausdorff dimension D satisfies the strong open set condition w.r.t. $\text{int } J$. Let the system of all $A_F(x, \epsilon)$ with $x \in F$, $\epsilon < \epsilon_0$, be a covariant neighborhood net with constants $a > 1$ and $\epsilon_0 > 0$. Let $b = \max(2a, \epsilon_0^{-1}|J|)$. Let $k \in \{0, 1, \dots, d\}$. If $k \leq d - 2$, we additionally suppose the regularity of parallel sets: Assumption 3.1. Recall

$$\Delta_{F,x,\delta}(R) := \frac{1}{|\ln \delta|} \int_{\delta}^{d(x,J^c)/b} \epsilon^{-k} C_k(F(\epsilon), A_F(x, \epsilon) \times R) \epsilon^{-1} d\epsilon.$$

Then for any fixed Borel set $R \subseteq S^{d-1}$ and $\mu_F \otimes \mathcal{H}_G$ -a.a. $(x, \phi) \in F \times G$ the following average limit exists

$$(3.2.2) \quad \lim_{\delta \rightarrow 0} \Delta_{F,x,\delta}(\phi(R)) = D_{C_k^{\text{frac}}|_F}(R)$$

provided the double integral in (3.2.1) converges absolutely. For $k \in \{d-1, d\}$ it always converges absolutely.

The assertion remains true (even without assuming integrability) if C_k is replaced everywhere with C_k^+ , C_k^- , or C_k^{var} , and accordingly $D_{C_k^{\text{frac}}|_F}$.

Remark 3.6. For the choice of A_F as in (3.1.3), the mapping $R \mapsto D_{C_k^{\text{frac}}|_F}(R)$ is the constant fibre of the disintegration of the associated k -th fractal curvature-direction measure over the Hausdorff measure, see section 3.4.

The case $k = d$ treats the Minkowski content and is formally included, but $D_{C_d^{\text{frac}}|_F}$ is always isotropic and does not provide geometric information beyond its total mass. Even that is redundant, because whenever $D < d$,

$$D_{C_d^{\text{frac}}|_F} = \frac{1}{d - D} D_{C_{d-1}^{\text{frac}}|_F},$$

and one side exists whenever the other one does, see [RW09, Theorems 4.5, 4.7] and Theorem 3.9.

Remark 3.7. Using (2.1.5), one can see that a sufficient (sharper) condition for the absolute convergence of the integral (and also for the uniform integrability in Theorem 3.9) is

$$(3.2.3) \quad \text{ess sup}_{\epsilon < \epsilon_0, (x, \phi) \in F \times G} \epsilon^{-k} C_k^{\text{var}}(F(\epsilon), A_F(x, \epsilon) \times \phi(R)) < \infty.$$

In case of polyconvex parallel sets $F(\epsilon)$ this is always satisfied. The modifications to [Win08, Lemma 5.3.2] are sketched in [RZ10, Remark 2.2.3].

In case of $k = d$ or $k = d - 1$, this also holds, see [RZ10, Remark 3.2.2].

More information on this condition can be found in [Win12].

3.3. Extended shift dynamical system and proof of the theorem. As an essential auxiliary tool for the proof we use the ergodic dynamical system $[W \times G, \nu \otimes \mathcal{H}_G, \theta]$ on the code space W and the compact group G generated by the φ_i , $i = 1, \dots, N$ from (2.1.4), for the skew-product shift operator $\theta : W \times G \rightarrow W \times G$ with $\theta(w_1 w_2 \dots, \phi) := (w_2 w_3 \dots, \varphi_{w_1}^{-1} \circ \phi)$. Recall φ_i (here the Rokhlin cocycle) is the orthogonal group component of S_i , \mathcal{H}_G is the unique normalized Haar measure on G , and $\mu_F(A) = \mathcal{H}^D(F \cap A) / \mathcal{H}^D(F)$. Let $P_{\mathbb{R}^d}$ be the projection onto the first component, and P_G onto the second.

Fact 3.8. $[W \times G, \nu \otimes \mathcal{H}_G, \theta]$ is ergodic. ([Gra95, Proposition 5.1])

According to (2.1.2), $[W \times G, \nu \otimes \mathcal{H}_G, \theta]$ induces the ergodic dynamical system $[F \times G, \mu_F \otimes \mathcal{H}_G, T]$, where the transformation $T : F \times G \rightarrow F \times G$ is defined for $\mu_F \otimes \mathcal{H}_G$ -a.a. (x, ϕ) by

$$T(x, \phi) := \left(S_j^{-1}x, \varphi_j^{-1} \circ \phi \right) \text{ if } x \in S_j(F), j = 1, \dots, N,$$

taking into regard that $\mu_F(S_i(F) \cap S_j(F)) = 0, i \neq j$. (More general references on this subject may be found, e.g., in Falconer [Fal97], Mauldin and Urbanski [MU03].) Recall we identify a.a. points in F with their coding sequences.

Now we will show the curvature located on the covariant neighborhood A_F is co-variant under the shift map. Note the transformed ϵ stays in the domain of definition: $\epsilon < b^{-1}d(x, (S_{x|l}J)^c)$ implies $\epsilon < r_{x|l}^{-1}\epsilon < \epsilon_0$, since $d(x, (S_{x|i}J)^c) = r_{x|i}d(P_{\mathbb{R}^d}T^l(x, \phi), J^c)$. From this and $A_F(x, \epsilon) \subseteq B(x, a\epsilon)$ we obtain for Lebesgue-a.a. $\epsilon, \mu_F \otimes \mathcal{H}_G$ -a.a. $(x, \phi) \in F \times G$, and $l \in \mathbb{N}$ satisfying the first condition the equalities

$$\begin{aligned} (3.3.1) \quad & C_k(F(\epsilon), A_F(x, \epsilon) \times \phi(R)) \\ &= C_k(F_{x|l}(\epsilon), A_F(x, \epsilon) \times \phi(R)) \\ &= C_k(F_{x|l}(\epsilon), S_{x|l}A_F(S_{x|l}^{-1}(x), r_{x|l}^{-1}\epsilon) \times \phi(R)) \\ &= r_{x|l}^k C_k(F(r_{x|l}^{-1}\epsilon), A_F(S_{x|l}^{-1}(x), r_{x|l}^{-1}\epsilon) \times (\varphi_{x|l}^{-1} \circ \phi(R))) \\ &= r_{x|l}^k C_k(F(r_{x|l}^{-1}\epsilon), A_F(P_{\mathbb{R}^d}T^l(x, \phi), r_{x|l}^{-1}\epsilon) \times P_G T^l(x, \phi)(R)). \end{aligned}$$

Here we have used the locality (2.2.4) of the curvature measure C_k , the covariance of the sets $A_F(x, \epsilon, R)$ (Def. 3.2), the scaling property (2.2.3) of C_k under similarities, and have expressed the inverse map of the IFS using the shift operator.

Measurability of $(x, \phi, \epsilon) \mapsto C_k(F(\epsilon), A_F(x, \epsilon) \times \phi(R))$ is proved the same way as in [RZ10, Lemma 2.3.1]. This works for measurability in x, ϵ for a fixed $\phi = \text{id}$. General ϕ can be recovered by concatenating with the (measurable) group action of ϕ .

Finally, the limit of the $\Delta_{F,x,\delta}$ measures

$$\lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{b^{-1}d(x, J^c)}^{\epsilon} \epsilon^{-k} C_k(F(\epsilon), A_F(x, \epsilon) \times \phi(R)) \epsilon^{-1} d\epsilon$$

can be checked using the methods of [RZ10, Section 2.3]. The main idea is to separate the integral into chunks the Birkhoff ergodic theorem can be applied to. The above computation shows each chunk is a summand in the ergodic average. Their $A_F(x, \epsilon)$ has to be replaced with our $A_F(x, \epsilon) \times \phi(R)$, $d\mu_F(x)$ with $d(\mu_F \otimes \mathcal{H}_G)(x, \phi)$, and their $d(T^i x, J^c)$ with $d(P_{\mathbb{R}^d}T^l(x, \phi), J^c)$.

The proof for the variation measures C_k^\pm works the same way. (The integrability assumption of the Birkhoff theorem can be replaced with positivity.) The extension of the limit to all $\delta \rightarrow 0$ is done by monotonicity arguments and by bounds on their $\delta(x, n(x, \delta'))/\delta'$ instead of their equation (28).

3.4. Main result 2: convergence of curvature-direction measures. Fractal curvature-direction measures exist under slightly stronger conditions. The local fractal curvatures $D_{C_k^{\text{frac}}|F}(R)$ from Theorem 3.5 play the role of (constant) fibre measures on F with respect to the normalized Hausdorff measure μ_F . This is in analogy to the case of a differentiable submanifold M of \mathbb{R}^d , where the (local) fibre measure on the sphere S^{d-1} over a point $x \in M$ is given by a symmetric polynomial of principal curvatures times signed unit mass atoms on the unit normals with footpoint x , and where the (global)

curvature-direction measure is the integral of these fibres with respect to the intrinsic Lebesgue measure on M .

Instead of adapting the proofs scattered over [RZ10, WZ10, Zäh11, Win08] in a routine way, we propose a shorter proof under weaker, sharp assumptions.

Following [RZ10], a Fubini argument is applied to the total curvature to move the Hausdorff measure on F , which is invariant under the shift on F , from inside the definition of the covariant neighborhood net, to the outermost integral (3.4.6). This connects the curvature-direction measure with the local fibre version and the shift dynamical system on F . Crucially unlike [RZ10], we already work with the measure at this stage instead of its total mass.

The group extension of the shift dynamical system accounts for the way pieces of the attractor F are rotated under the IFS. This is not neutralized by integrating over F alone, as done in the Fubini expression, so we need a convergence result for unrotated fibre measures. Using the Birkhoff ergodic theorem, Theorem 3.5 gave us a result on fibres rotated in almost any way. Corollary 3.10 improves this to unrotated ones, at the price of weak convergence.

The dynamical system shrinks the covariant neighborhood net of each point at its own speed, whereas the parallel set width has to be shrunk globally. This discrepancy introduces a uniform integrability condition, which we show to be necessary and sufficient for the convergence of fractal curvature-direction measures to our formula (Proposition 3.12). See also Remark 3.7 for an “easier”, sufficient condition.

Finally, Corollary 3.14 touches upon whether the similarity dimension is always equal to the curvature scaling dimension. If the fractal curvature vanishes trivially, then the curvature of the parallel sets F_ϵ is concentrated near the boundary ∂J of the (SOSC) open set.

Recall $D_{C_k^{\text{frac}}|_F}$ is defined by (3.2.1).

Theorem 3.9. (*Fractal curvature-direction measure*) Suppose that the self-similar set F in \mathbb{R}^d with Hausdorff dimension D satisfies the strong open set condition w.r.t. $\text{int } J$. Let the system of all $A_F(x, \epsilon)$ with $x \in F$, $\epsilon < \epsilon_0$, be the covariant neighborhood net given in (3.1.3) with constant $a = 2c_F^{-1/D}$, where $c_F \leq 1$ fulfills (2.1.6). Let $b = \max(2a, \epsilon_0^{-1}|J|)$.

Let $k \in \{0, 1, \dots, d\}$. If $k \leq d - 2$, we additionally suppose the regularity of parallel sets, Assumption 3.1, and assume the function

$$(3.4.1) \quad F \ni x \mapsto \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{-k} C_k^{\text{var}}(F(\epsilon), B(x, a\epsilon) \times S^{d-1}) \epsilon^{-1} d\epsilon$$

is uniformly \mathcal{H}^D -integrable for $0 < \delta < \epsilon_0$.

Then the following limit exists in the weak sense:

$$(3.4.2) \quad C_k^{\text{frac}}(F, \cdot) := \lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{D-k} C_k(F(\epsilon), \cdot) \epsilon^{-1} d\epsilon.$$

The limit is a product and is uniquely specified by

$$C_k^{\text{frac}}(F, B \times R) = \mathcal{H}^D(F \cap B) D_{C_k^{\text{frac}}|_F}(R)$$

for Borel sets $B \subseteq \mathbb{R}^d$, $R \subseteq S^{d-1}$.

The assertion is true for C_k^+ , C_k^- , and C_k^{var} , see Proposition 3.12 below. A sufficient condition for uniform integrability was discussed in Remark 3.7. Under that sufficient

condition, there is a more convenient formula for the total mass of $D_{C_k^{\text{frac}}|F}$, see [Win08, Theorem 2.3.6] and [Zäh11, Theorem 2.3.8]. We conjecture it also works in our case.

Proof. Since the set $F(\epsilon_0) \times S^{d-1}$ is compact, any continuous function on it can be approximated in norm with product type functions. Linearity and the fact $A_F(x, \epsilon) \subseteq B(x, a\epsilon)$ then reduce the problem to Proposition 3.12 below. In case $k = d$ or $k = d - 1$, uniform integrability is due to (3.2.3). \square

The following corollary to Theorem 3.5 strengthens it to hold for all elements of G instead of \mathcal{H}_G -almost all, at the price of weak convergence. Geometrically, it means $D_{C_k^{\text{frac}}|F}^\pm$ is the “distribution” of normal rays on the fractal, weighted by the k -th higher order mean curvature.

Corollary 3.10. *Assume the conditions of Theorem 3.9 except uniform integrability of (3.4.1), and let $\pm \in \{+, -\}$. Write*

$$(3.4.3) \quad \Delta_{F,x,\delta}^\pm(R) := \frac{1}{|\ln \delta|} \int_{\delta}^{d(x, J^c)/b} \epsilon^{-k} C_k^\pm(F(\epsilon), A_F(x, \epsilon) \times R) \epsilon^{-1} d\epsilon, \quad R \subseteq S^{d-1} \text{ Borel}.$$

Then for μ_F -a.a. $x \in F$, the following limit exists and does not depend on x :

$$(3.4.4) \quad \lim_{\delta \rightarrow 0} \Delta_{F,x,\delta}^\pm(g) = D_{C_k^{\text{frac}}|F}^\pm(g), \text{ for continuous } g : S^{d-1} \rightarrow [0, 1].$$

Proof. Firstly, $\Delta_{F,x,\delta}^\pm$ and $D_{C_k^{\text{frac}}|F}^\pm$ are Borel measures on S^{d-1} : Measurability of the integrand is covered in Section 3.3. To show σ -additivity using the defining formulas (3.2.1), (3.4.3), we combine the measure property of $C_k^\pm(F(\epsilon), \cdot)$, and the monotone convergence lemma.

Next, we will show convergence of $\Delta_{F,x,\delta}^\pm \circ \phi^{-1}$ for almost all $x \in F$, $\phi \in G$. Let \mathcal{R} be a countable system of open sets that generates the topology of S^{d-1} . We will assume $S^{d-1} \in \mathcal{R}$, to help with the total masses. Set

$$\Phi := \left\{ \phi \in G : \lim_{\delta \rightarrow 0} \Delta_{F,x,\delta}^\pm(\phi R) = D_{C_k^{\text{frac}}|F}^\pm(R) \text{ for all } R \in \mathcal{R}, \mu_F\text{-a.a. } x \in F \right\}.$$

Φ has full \mathcal{H}_G measure since \mathcal{R} is countable, so Φ is not empty (Theorem 3.5).

Consider the case $D_{C_k^{\text{frac}}|F}^\pm$ has finite mass. A generator of the topology is a convergence-determining class. ([Bil99, Theorem 2.3] also holds for sequences of finite measures if the total masses converge.) Consequently, we have weak convergence: for all $\phi \in \Phi$, almost all $x \in F$, and all continuous $g : S^{d-1} \rightarrow [0, 1]$,

$$\lim_{\delta \rightarrow 0} \Delta_{F,x,\delta}^\pm(g \circ \phi^{-1}) = D_{C_k^{\text{frac}}|F}^\pm(g).$$

Since $g \circ \phi$ is also continuous, we may substitute it for g ,

$$\lim_{\delta \rightarrow 0} \Delta_{F,x,\delta}^\pm(g) = D_{C_k^{\text{frac}}|F}^\pm(g \circ \phi).$$

As a Haar integral, $D_{C_k^{\text{frac}}|F}^\pm$ is invariant under ϕ :

$$\lim_{\delta \rightarrow 0} \Delta_{F,x,\delta}^\pm(g) = D_{C_k^{\text{frac}}|F}^\pm(g).$$

In case of infinite $D_{C_k^{\text{frac}}|F}^\pm$, both it and the map $g \mapsto \liminf_{\delta \rightarrow 0} \Delta_{F,x,\delta}^\pm(g)$ still are Φ -invariant, and Φ is dense in G . A finite covering argument shows both are infinite on every open set $R \neq \emptyset$ and (any level set of) every non-vanishing g , thus the assertion. \square

This well-known fact will compensate that the supports of μ_F and $C_k(F(\epsilon), \cdot)$ are close but not identical:

Lemma 3.11. *Let X be a metric space, $g : X \times (0, \infty) \rightarrow [0, \infty)$, $g_0 : X \rightarrow [0, \infty)$, and let f be a continuous and bounded function $f : X \rightarrow \mathbb{R}$, and $x \in X$ such that $f(x) \neq 0$. The following are equivalent, including the existence of the limits:*

- (1) $f(x) \lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\delta_0} g(x, \epsilon) \epsilon^{-1} d\epsilon = f(x) g_0(x),$
- (2) $\lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\delta_0} \left(\min_{y \in B(x, \epsilon)} f(y) \right) g(x, \epsilon) \epsilon^{-1} d\epsilon = f(x) g_0(x).$

Proof. Note $\min_{y \in B(x, \epsilon)} |f(y)| > 0$ for sufficiently small ϵ . Denote by ω_f the modulus of continuity. Since $\min_{B(x, \epsilon)} f \leq f(x) \leq \min_{B(x, \epsilon)} f + \omega_f(\epsilon)$, it is enough to show

$$\limsup_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\delta_0} \omega_f(\epsilon) g(x, \epsilon) \epsilon^{-1} d\epsilon = 0.$$

Given any $\eta > 0$, there is a δ_1 such that $\omega_f(\epsilon) \leq \eta \left| \min_{B(x, \epsilon)} f \right| \leq \eta |f(x)|$ whenever $\epsilon < \delta_1$. We split the above integral at δ_1 . The fact $g_0(x)$ is finite implies $\int_{\delta_1}^{\delta_0} g(x, \epsilon) \epsilon^{-1} d\epsilon < \infty$, so that part of the limsup becomes zero. The other part is at most $\eta g_0(x) |f(x)|$. \square

Proposition 3.12. *Assume the conditions of Theorem 3.9 except uniform integrability of (3.4.1), and let $\pm \in \{+, -\}$. Then for all continuous $f : \mathbb{R}^d \rightarrow [0, \infty)$ and $g : S^{d-1} \rightarrow [0, \infty)$,*

$$\liminf_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{D-k} C_k^{\pm}(F(\epsilon), f \otimes g) \epsilon^{-1} d\epsilon \geq \left(\sum_{j=1}^N r_j^D |\ln r_j| \right)^{-1} \mathcal{H}^D(1_F f) D_{C_k^{\text{frac}}|_F}^{\pm}(g).$$

The following are equivalent:

- (1) The lower limit exists as a limit, and we have equality.
- (2) The expression

$$(x, \phi) \mapsto \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{-k} C_k^{\pm}(F(\epsilon), 1_{A_F(x, \epsilon)} \otimes (g \circ \phi^{-1})) \epsilon^{-1} d\epsilon$$

is uniformly $\mu_F \otimes \mathcal{H}_G$ -integrable for $0 < \delta < \epsilon_0$, or both sides of the assertion are infinite, or f is zero on F .

The integral in (2) differs from $\Delta_{F, x, \delta}(g \circ \phi^{-1})$ only by its upper boundary.

Proof. The left side of the assertion will be assumed finite for the entire proof.

We will reduce the problem to Theorem 3.5 and the previous corollary, taking into regard the relationship

(3.4.5)

$$\epsilon^D \int_{\mathbb{R}^d \times S^{d-1}} f(z) g(n) dC_k^{\pm}(F(\epsilon), (z, n)) \geq \int_F \left(\min_{B(x, a\epsilon)} f \right) C_k^{\pm}(F(\epsilon), 1_{A_F(x, \epsilon)} \otimes g) d\mathcal{H}^D(x)$$

for a.a. $\epsilon < \epsilon_0$, with equality if f is a constant. The sets A_F were defined in (3.1.3) to make this possible: ($R \subseteq S^{d-1}$ Borel)

$$\begin{aligned}
 (3.4.6) \quad & \epsilon^D \int_{F(\epsilon) \times R} f(z) dC_k^\pm(F(\epsilon), (z, n)) \\
 &= \int_{F(\epsilon) \times R} f(z) \mathcal{H}^D(F \cap B(z, \rho_F(z, \epsilon))) dC_k^\pm(F(\epsilon), (z, n)) \\
 &= \int_F \int_{F(\epsilon) \times \text{id } R} f(z) \mathbf{1}(|x - z| \leq \rho_F(z, \epsilon)) dC_k^\pm(F(\epsilon), (z, n)) d\mathcal{H}^D(x) \\
 &\geq \int_F \left(\min_{B(x, a\epsilon)} f \right) \int_{F(\epsilon) \times \text{id } R} \mathbf{1}(|x - z| \leq \rho_F(z, \epsilon)) dC_k^\pm(F(\epsilon), (z, n)) d\mathcal{H}^D(x) \\
 &= \int_F \left(\min_{z \in B(x, a\epsilon)} f(z) \right) C_k^\pm(F(\epsilon), A_F(x, \epsilon) \times R) d\mathcal{H}^D(x).
 \end{aligned}$$

We were allowed to apply Fubini because the integrand is positive and all measures σ -finite. Approximating with staircase functions yields (3.4.5). Then we get

$$\begin{aligned}
 (3.4.7) \quad & \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{D-k} \int_{F(\epsilon)} f(z) g(n) dC_k^\pm(F(\epsilon), (z, n)) \epsilon^{-1} d\epsilon \\
 &\geq \int_F \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \left(\min_{B(x, a\epsilon)} f \right) \epsilon^{-k} C_k^\pm(F(\epsilon), 1_{A_F(x, \epsilon)} \otimes g) \epsilon^{-1} d\epsilon d\mathcal{H}^D(x).
 \end{aligned}$$

Fatou's lemma lets us take the lower limit as $\delta \rightarrow 0$ under this integral. The liminf on the left side was assumed finite, so the integrand on the right is finite almost everywhere. Next, Lemma 3.11 permits us to replace $(\min_{B(x, a\epsilon)} f)$ with $f(x)$, except in case $f(x) = 0$,

where the minimum already agrees with $f(x)$.

$$\begin{aligned}
& \liminf_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{D-k} \int_{F(\epsilon) \times S^{d-1}} f(z) g(n) dC_k^{\pm}(F(\epsilon), (z, n)) \epsilon^{-1} d\epsilon \\
& \geq \int_F \liminf_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \left(\min_{B(x, a\epsilon)} f \right) \epsilon^{-k} C_k^{\pm}(F(\epsilon), 1_{A_F(x, \epsilon)} \otimes g) \epsilon^{-1} d\epsilon d\mathcal{H}^D(x) \\
& = \int_F \lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \left(\min_{B(x, a\epsilon)} f \right) \epsilon^{-k} C_k^{\pm}(F(\epsilon), 1_{A_F(x, \epsilon)} \otimes g) \epsilon^{-1} d\epsilon d\mathcal{H}^D(x) \\
& = \int_F \lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} f(x) \int_{\delta}^{\epsilon_0} \epsilon^{-k} C_k^{\pm}(F(\epsilon), 1_{A_F(x, \epsilon)} \otimes g) \epsilon^{-1} d\epsilon d\mathcal{H}^D(x) \\
& = D_{C_k^{\text{frac}}|_F}^{\pm}(g) \int_F f(x) d\mathcal{H}^D(x)
\end{aligned}$$

Reading backwards, Theorem 3.5 guarantees that the lower limit is a limit and Lemma 3.11 indeed applicable. We have proved the asserted inequality.

Now assume uniform integrability due to assertion (2). We get equality in Fatou's lemma. In case f is a constant, (3.4.7) becomes an equality, too. That proves assertion (1) first for a constant f . The general case follows from positivity and equal mass. Conversely, equality in Fatou's lemma implies L_1 convergence, which in turn implies uniform integrability ([Doo94, Chapters 6.8 and 6.18]). \square

Remark 3.13. The convergence result also holds on measurable functions $f \otimes g$ of type

$$(x, n) \mapsto f(x) 1_R(n), \quad f \text{ continuous, } R = \varphi_i R \subseteq S^{d-1}, i = 1, \dots, N,$$

i.e. R is a totally invariant Borel set. (This is stronger than convergence on all continuous functions.) Instead of setting $\phi = \text{id} \in G$ in the proof by invoking Corollary 3.10, we can exploit $g \circ \phi = g$ using Theorem 3.5 directly. Simply replace $d\mathcal{H}^D(x)$ with $d(\mathcal{H}^D \otimes \mathcal{H}_G)(x, \phi)$, \int_F with $\int_{F \times G}$, and id with ϕ .

In some degenerate cases, the similarity dimension D can be an inappropriate rescaling, and $C_k^{\text{frac}}(F, \cdot)$ can vanish trivially. One such example is the unit cube $[0, 1]^d \subseteq \mathbb{R}^d$, which is generated by 2^d similarites with contraction ratio $1/2$ and fixed points in the corners. One of both variations can vanish even for “true” fractals, for example $C_0^{+, \text{frac}}$ of the Sierpinski gasket. (See [Win08] for further discussion.) Such a situation can be detected by the concentration of the curvature near ∂J as $\epsilon \rightarrow 0$, i.e. away from μ_F -most of F .

Corollary 3.14. *In the situation of the proposition, if*

$$\lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_{\delta}^{\epsilon_0} \epsilon^{D-k} C_k^{\pm}(F(\epsilon)) \epsilon^{-1} d\epsilon = 0,$$

then

$$C_k^{\pm}(F(\epsilon), \overline{J^c}(\epsilon c)^c \times S^{d-1}) = 0$$

for almost all $\epsilon < \epsilon_0$, where

$$c := \max \left\{ 4 c_F^{-\frac{1}{D}}, |J| \mathcal{H}^D(F)^{-\frac{1}{D}} \right\} - C_F^{-\frac{1}{D}},$$

and C_F, c_F are constants from (2.1.6).

Remark 3.15. Enlarging the σ -algebra can increase the mass of the absolute variation measure. In principle, our integrability assumption on $C_k(F(\epsilon), \cdot)$ as a measure in $\mathbb{R}^d \times S^{d-1}$ could be stronger than assumptions on $C_k(F(\epsilon), \cdot \times S^{d-1})$ in \mathbb{R}^d . [RZ10] worked with the latter. However, we know no examples of parallel sets $F(\epsilon)$ where this matters. For example, let $K := \{x \in \mathbb{R}^2 : |x| = 1\}$ be the unit circle. The set of outpointing normals carries the positive C_0 mass, but they always share a footpoint with an inpointing normal carrying negative mass.

$$\begin{aligned} \sup_{A \subseteq \mathbb{R}^2 \text{ Borel}} C_0(K, A \times S^{d-1}) &= 0, \\ \sup_{A \subseteq \mathbb{R}^2 \times S^{d-1} \text{ Borel}} C_0(K, A) &= 1. \end{aligned}$$

Fact 3.16. ([Rat08, Lemma 3]) *If $F(\epsilon)$ is a finite union of convex sets for some (and therefore all) $\epsilon > 0$, then for $k = 1, \dots, d$,*

$$\sup_{A \subseteq \mathbb{R}^2 \text{ Borel}} C_k(F(\epsilon), A \times S^{d-1}) = \sup_{A \subseteq \mathbb{R}^2 \times S^{d-1} \text{ Borel}} C_k(F(\epsilon), A),$$

i.e. for almost all $x \in F(\epsilon)$, the normal cone over x contributes either purely positive or purely negative mass.

The lemma also holds for arbitrary compact sets F if ϵ is large enough, see the proof of [Zäh11, Theorem 4.1].

4. EXAMPLES

If F is the classical Sierpinski triangle, both $D_{C_1^{\text{frac}}|F}$ and $D_{C_0^{\text{frac}}|F}$ are a constant multiples of their classical counterpart for an equilateral triangle, i.e. $D_{C_1^{\text{frac}}|F}$ has equal atoms on the normals onto the three sides, and $D_{C_0^{\text{frac}}|F}$ is a constant times the uniform distribution on S^1 . ($D_{C_2^{\text{frac}}|F}$ is always rotation invariant for sets F in \mathbb{R}^2 .)

Let $F \subseteq \mathbb{R}^2$ instead be the modified Sierpinski gasket shown in Figure 4.2. It is generated by the following IFS: S_i , $i = 1, 2, 3$ are the same as for the Sierpinski gasket. They contract by $r_i = 1/2$, rotate by $2\pi i/3 \bmod 2\pi$, and move the upper vertex onto the left, right, or upper vertex, respectively. The last similarity, S_4 , contracts by $r_4 = 1/4$, rotates by $\alpha \bmod 2\pi$, and has its fixed point $(1/2, \sqrt{3}/6)$ in the center. (F is connected only if α is an integer multiple of $2\pi/3$.)

The parallel sets are regular due to polyconvexity. The Open Set Condition is satisfied with $J := \text{cl conv } F$.

The compact group G generated by the rotational components is

$$G = \begin{cases} S^1 & \text{if } \frac{\alpha}{2\pi} \notin \mathbb{Q}, \\ \left\{ k\alpha + l \frac{2\pi}{3} \bmod 2\pi : k, l \in \mathbb{Z} \right\} & \text{if } \frac{\alpha}{2\pi} \in \mathbb{Q}. \end{cases}$$

$D_{C_1^{\text{frac}}|F}$ exhibits only the symmetry demanded by G -invariance. Identifying S^1 and $SO(2)$ such that $0 \approx \text{id}$ points downwards, $D_{C_1^{\text{frac}}|F}$ is a (positive) multiple of the Haar measure on G . (Fine) approximation sets obtained by applying the IFS finitely often to J have a

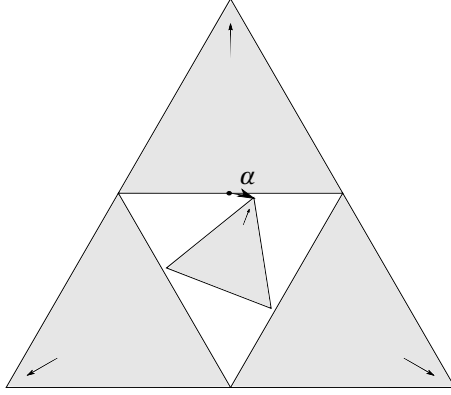
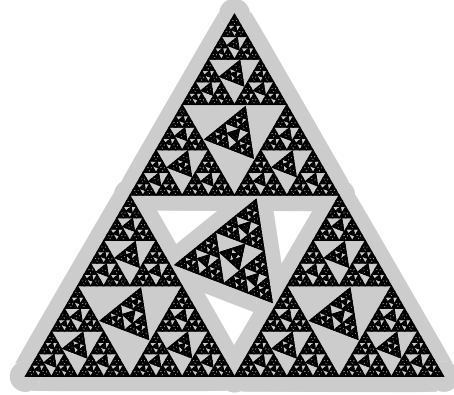


FIGURE 4.1. IFS: images of convex hull

FIGURE 4.2. Attractor F and parallel set $F(\epsilon)$ (shaded)

C_1 measure with the same support. By contrast, $D_{C_0^{\text{frac}}|F}$ is always a (negative) multiple of the Haar measure on S^1 .

If $\alpha = 0$, both $D_{C_0^{\text{frac}}|F}$ and $D_{C_1^{\text{frac}}|F}$ agree with their classical counterpart (area measure) for a convex, equilateral triangle.

Since $D_{C_k^{\text{frac}}|F}$ is non-null, the fractal curvature-direction measures are the product of $D_{C_k^{\text{frac}}|F}$ with the normalized Hausdorff measure on F : $C_k^{\text{frac}}(F, \cdot) = D_{C_k^{\text{frac}}|F} \otimes \mu_F$.

Finally, the interpretation as fractal curvatures is justified by the fact they do not vanish trivially, i.e. D is the appropriate choice of rescaling exponent. This follows either from an explicit computation or from [Win08, Theorem 2.3.8].

REFERENCES

- [BG92] Christoph Bandt and Siegfried Graf, *Self-similar sets. VII. A characterization of self-similar fractals with positive Hausdorff measure*, Proc. Amer. Math. Soc. **114** (1992), no. 4, 995–1001.
- [Bil99] Patrick Billingsley, *Convergence of probability measures*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999, A Wiley-Interscience Publication.
- [Doo94] J. L. Doob, *Measure theory*, Graduate Texts in Mathematics, vol. 143, Springer-Verlag, New York, 1994.
- [Fal95] K. J. Falconer, *On the Minkowski measurability of fractals*, Proc. Amer. Math. Soc. **123** (1995), no. 4, 1115–1124.
- [Fal97] Kenneth Falconer, *Techniques in fractal geometry*, John Wiley & Sons Ltd., Chichester, 1997.
- [Fed59] Herbert Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.
- [Fu85] Joseph Howland Guthrie Fu, *Tubular neighborhoods in Euclidean spaces*, Duke Math. J. **52** (1985), no. 4, 1025–1046.
- [Gat00] Dimitris Gatzouras, *Lacunarity of self-similar and stochastically self-similar sets*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 1953–1983.
- [Gra95] Siegfried Graf, *On Bandt's tangential distribution for self-similar measures*, Monatsh. Math. **120** (1995), no. 3–4, 223–246.
- [Hut81] John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747.
- [KK10] Marc Kesseböhmer and Sabrina Kombrink, *Fractal curvature measures and Minkowski content for one-dimensional self-conformal sets*, Arxiv 1012.5399.
- [LP93] Michel L. Lapidus and Carl Pomerance, *The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums*, Proc. London Math. Soc. (3) **66** (1993), no. 1, 41–69.

- [MU03] R. Daniel Mauldin and Mariusz Urbański, *Graph directed Markov systems*, Cambridge Tracts in Mathematics, vol. 148, Cambridge University Press, Cambridge, 2003, Geometry and dynamics of limit sets.
- [Rat08] J. Rataj, *Convergence of total variation of curvature measures*, Monatsh. Math. **153** (2008), no. 2, 153–164.
- [RW09] Jan Rataj and Steffen Winter, *On volume and surface area of parallel sets*, 2009, Arxiv 0905.3279v2.
- [RZ03] J. Rataj and M. Zähle, *Normal cycles of Lipschitz manifolds by approximation with parallel sets*, Differential Geom. Appl. **19** (2003), no. 1, 113–126.
- [RZ05] ———, *General normal cycles and Lipschitz manifolds of bounded curvature*, Ann. Global Anal. Geom. **27** (2005), no. 2, 135–156.
- [RZ10] ———, *Curvature densities of self-similar sets*, Arxiv 1009.6162.
- [Sch80] Rolf Schneider, *Parallelmengen mit Vielfachheit und Steiner-Formeln*, Geom. Dedicata **9** (1980), no. 1, 111–127.
- [Sch94] Andreas Schief, *Separation properties for self-similar sets*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 111–115.
- [Win08] Steffen Winter, *Curvature measures and fractals*, Dissertationes Math. (Rozprawy Mat.) **453** (2008), 66.
- [Win12] S. Winter, *Curvature Bounds for Neighborhoods of Self-Similar Sets*, Comment. Math. Univ. Carolin. (2012), Arxiv 1010.2032.
- [WZ10] Steffen Winter and Martina Zähle, *Fractal curvature measures for self-similar sets*, 2010, Arxiv 1007.0696.
- [Zäh86] M. Zähle, *Integral and current representation of Federer’s curvature measures*, Arch. Math. (Basel) **46** (1986), no. 6, 557–567.
- [Zäh11] ———, *Lipschitz-Killing curvatures of self-similar random fractals*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2663–2684.

E-mail address: tilman.rothe@uni-jena.de, martina.zaehle@uni-jena.de

FRIEDRICH SCHILLER UNIVERSITY JENA